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WEAKLY MIXING REMARKS

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Weakly mixing remarks

by

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ABSTRACT

We study homomorphisms of minimal transformation groups that admit relatively invariant measures, especially with respect to the equicontinuous structure relation and weak disjointness. In particular we prove that for an open RIM extension the equicontinuous structure relation equals the regionally proximal relation.

KEY WORDS & PHRASES: *Minimal transformation group, ergodicity, Relatively Invariant Measure, equicontinuous structure relation*

1. INTRODUCTION

Although we assume basic knowledge about topological dynamics as can be found in [G₁], [B] we will review some basic definitions. A *topological transformation group* (ttg) is a triple (T, X, π) , where T is a topological group, X a compact T_2 space and $\pi: T \times X \rightarrow X$ is a continuous map such that $\pi(e, x) = x$ and $\pi(s, \pi(t, x)) = \pi(st, x)$ for all $x \in X$, $t, s \in T$. We will fix the group and drop the action symbol. A subset $A \subseteq X$ is called *invariant* if $TA = A$ and X is called *minimal* (*ergodic*) if the only nonempty closed invariant subset of X (with non-empty interior) is X itself.

A continuous surjection $\phi: X \rightarrow Y$ between two ttg's is called a *homomorphism of ttg's*, or an *extension* if $\phi(tx) = t\phi(x)$ for all $t \in T$, $x \in X$. Any homomorphism induces a closed invariant equivalence relation $R_\phi = \{(x_1, x_2) \in X \times X \mid \phi(x_1) = \phi(x_2)\}$ on X .

Let \mathcal{U}_X denote the unique uniform structure on X , then we define $P_\phi = \bigcap \{T\alpha \cap R_\phi \mid \alpha \in \mathcal{U}_X\}$, $Q_\phi = \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\}$ the *proximal* and *regionally proximal* relation of ϕ ; and E_ϕ the *equicontinuous structure relation* of ϕ is defined to be the smallest closed invariant equivalence relation that contains Q_ϕ . One of the major problems in topological dynamics is to determine E_ϕ . VEECH [V₂], showed that $E_\phi = Q_\phi$ if the almost periodic points (points with minimal orbitclosure) are dense in R_ϕ . McMAHON [M] and McMAHON and WU [MW'80] proved related results with totally different methods. We shall use those methods to prove that $E_\phi = Q_\phi$ in case ϕ is an open RIM extension. We call $\phi: X \rightarrow Y$ a *RIM extension* if there exists a homomorphism $\lambda: Y \rightarrow M(X)$ (into) such that $\hat{\phi} \circ \lambda: Y \rightarrow M(X) \rightarrow M(Y)$ equals id_Y , where $M(X)$ is the set of Borel probability measures with the weak star topology and $\hat{\phi}: M(X) \rightarrow M(Y)$ is the map induced by ϕ (for more details see [G₂]).

We will also be concerned with the question: when are two homomorphisms $\phi: X \rightarrow Y$ and $\psi: Z \rightarrow Y$ of minimal ttg's *weakly disjoint*, i.e. when is $R_{\phi\psi} = \{(x, z) \in X \times Z \mid \phi(x) = \psi(z)\}$ an ergodic ttg. (Notation: $\phi \perp \psi$). We call ϕ *weakly mixing* if $\phi \perp \phi$ and a minimal ttg X is *weakly mixing* if $\phi: X \rightarrow 1$ is weakly mixing.

An interesting result is: if $\phi: X \rightarrow Y$ is a RIM extension of metric minimal ttg's without non-trivial almost periodic factors, then ϕ is weakly disjoint from every open extension of minimal ttg's that values in Y .

2. ERGODIC POINTS

Let $\phi: X \rightarrow Y$ be a homomorphism of minimal ttg's and $n \in \mathbb{N}$, $n \geq 2$. We call $x \in X$ a ϕ - n -local ergodic point if for every open subset $W \subseteq X$ there exists a set U open in $\phi^{\leftarrow}\phi(x)$ such that $U \supseteq E_{\phi}[x]$ and for open (in $\phi^{\leftarrow}\phi(x)$) sets V_1, \dots, V_n in U we have that $T(V_1 \times \dots \times V_n) \cap \prod_n W \neq \emptyset$. If U can be chosen to be $\phi^{\leftarrow}\phi(x)$, we call x ϕ - n -ergodic and clearly every $x' \in \phi^{\leftarrow}\phi(x)$ is ϕ - n -ergodic iff x is. If x is ϕ - n -(local) ergodic for all $n \in \mathbb{N}$, $n \geq 2$, we call x ϕ -(local) ergodic.

Obviously every ϕ - n -ergodic point is ϕ - n -local ergodic and if $E_{\phi} = R_{\phi}$ the converse is true.

If $\phi: X \rightarrow 1$ we skip the prefix ϕ in the above definitions. Note that if x is ϕ - n -(local) ergodic then tx is for all $t \in T$.

1. PROPOSITION. Let $\phi: X \rightarrow Y$ be a homomorphism of minimal ttg's.

If $x \in X$ is ϕ -2-local ergodic then $Q_{\phi}[x] = E_{\phi}[x]$.

If $x \in X$ is ϕ -2-ergodic then $Q_{\phi}[x] = R_{\phi}[x] = \phi^{\leftarrow}\phi(x)$.

PROOF. Choose $(x, x') \in R_{\phi}(E_{\phi})$ and $\alpha \in U_X$. Choose $\beta \in U_X$ with $\beta = \beta^{-1}$ and $\beta \circ \beta \subseteq \alpha$ then $T(\beta(x) \times \beta(x)) \subseteq T\alpha$. Choose U for $W = \beta(x)$ as in the definition. For every neighbourhood $V \times V'$ of (x, x') in $\phi^{\leftarrow}\phi(x) \times \phi^{\leftarrow}\phi(x)$ (in $U \times U$) we have $V \times V' \cap T(\beta(x) \times \beta(x)) \neq \emptyset$, so $(x, x') \in T\alpha \cap \phi^{\leftarrow}\phi(x) \times \phi^{\leftarrow}\phi(x) \subseteq T\alpha \cap R_{\phi}$, i.e. $(x, x') \in Q_{\phi}$. \square

2. COROLLARY. Let $\phi: X \rightarrow Y$ be a homomorphism of minimal ttg's. If there is a ϕ -2-ergodic point $x \in X$ then $E_{\phi} = R_{\phi}$.

PROOF. Since every x' in $\phi^{\leftarrow}\phi(x)$ is a ϕ -2-ergodic point it follows that $\phi^{\leftarrow}\phi(x) \times \phi^{\leftarrow}\phi(x) = \phi^{\leftarrow}\phi(x) \times \phi^{\leftarrow}\phi(x) \cap Q_{\phi} \subseteq E_{\phi}$. But then $\theta: X/E_{\phi} \rightarrow Y$ is almost one to one and almost periodic so θ is a homeomorphism and $X/E_{\phi} \cong Y$, $E_{\phi} = R_{\phi}$. \square

For the following we need to remember that $\phi: X \rightarrow Y$ is open iff for all $y \in Y$, $x \in \phi^{\leftarrow}(y)$ and for any net $y_i \rightarrow y$ we can choose $x_i \in \phi^{\leftarrow}(y_i)$ with $x_i \rightarrow x$.

3. PROPOSITION. Let $\phi: X \rightarrow Y$ be a homomorphism of minimal ttg's such that $\kappa: X \rightarrow X/E_\phi$ is open. Suppose there exists a $y \in Y$ such that x is ϕ -2-local ergodic for all $x \in \phi^{\leftarrow}(y)$. Then $E_\phi = Q_\phi$.

PROOF. Since x is ϕ -2-local ergodic for every $x \in \phi^{\leftarrow}(y)$ it follows from Prop.1 that $Q_\phi \cap \phi^{\leftarrow}(y) \times \phi^{\leftarrow}(y) = E_\phi \cap \phi^{\leftarrow}(y) \times \phi^{\leftarrow}(y)$. Let $z = \kappa(x) \in X/E_\phi$. Choose $(x_1, x_2) \in E_\phi$ and let $z_0 = \kappa(x_1) = \kappa(x_2)$. Choose a net $\{t_i\}$ in T with $t_i z \rightarrow z_0$. Since κ is open we can choose x_1^i and x_2^i in $\kappa^{\leftarrow}(z)$ such that $t_i(x_1^i, x_2^i) \rightarrow (x_1, x_2)$. Now $(x_1^i, x_2^i) \in E_\phi \cap \phi^{\leftarrow}(y) \times \phi^{\leftarrow}(y) \subseteq Q_\phi$ and so $(x_1, x_2) \in \overline{Q_\phi} = Q_\phi$. \square

4. COROLLARY. Let $\phi: X \rightarrow Y$ be an open homomorphism of minimal ttg's. If there is an $x \in X$ that is ϕ -2-ergodic then $Q_\phi = R_\phi$.

PROOF. By Cor.2 $E_\phi = R_\phi$ so, with notation as in 3 $\kappa = \phi$ is open. Since x' is ϕ -2-ergodic for all $x' \in \phi^{\leftarrow}\phi(x)$ it follows from Prop.3 that $Q_\phi = E_\phi = R_\phi$. \square

5. COROLLARY. Let X be minimal, then $Q_X = X \times X$ iff there exists a 2-ergodic point ($Q_X = Q_\phi$ with $\phi: X \rightarrow 1$).

PROOF. The "if" part is just Cor.4. Choose U and V open in X and $(x_1, x_2) \in U \times V$, then $(x_1, x_2) \in Q_X$ so $U \times V \cap T\alpha \neq \emptyset$ for any $\alpha \in U_X$. Clearly for any $W \subseteq X$ open, there is an $\alpha \in U_X$ with $\alpha \subseteq T(W \times W)$ and so $T\alpha \subseteq T(W \times W)$. \square

Let $\phi: X \rightarrow Y$ be a homomorphism of ttg's and $n \in \mathbb{N}$, $n \geq 2$. A point $x_0 \in X$ is called a P_ϕ^n -point if $\{(x_1, \dots, x_n) \in (\phi^{\leftarrow}\phi(x_0))^n \mid \overline{T(x_1, \dots, x_n)} \cap \Delta_X^n \neq \emptyset\}$ is dense in $(\phi^{\leftarrow}\phi(x_0))^n$. Clearly, if ϕ is proximal then every x is a P_ϕ^n point for all $n \in \mathbb{N}$, $n \geq 2$.

7. PROPOSITION. Let $\phi: X \rightarrow Y$ be a homomorphism of ttg's and $n \in \mathbb{N}$, $n \geq 2$.

a. If X is minimal, then every P_ϕ^n -point is a ϕ - n -ergodic point.

b. If x_0 has a countable neighbourhood base \mathcal{W}_{x_0} for some $x_0 \in X$, then every ϕ - n -ergodic point is a P_ϕ^n -point.

In particular if X is metric and minimal, then the ϕ - n -ergodic points are just the P_ϕ^n -points.

PROOF. a Let $x \in X$ be a P_ϕ^n -point. Choose $W \subseteq X$ open and U_1, \dots, U_n open in $\phi^{\leftarrow} \phi(x)$. Then, since X is minimal, $\Delta_X^n \subseteq T(W \times \dots \times W)$ (n -times). Since $U_1 \times \dots \times U_n$ is open in $(\phi^{\leftarrow} \phi(x))^n$ there is a $(x_1, \dots, x_n) \in U_1 \times \dots \times U_n$ with $\Delta_X^n \cap \overline{T(x_1, \dots, x_n)} \neq \emptyset$. So $\Delta_X^n \subseteq \overline{T(U_1 \times \dots \times U_n)}$ and $\overline{T(U_1 \times \dots \times U_n)} \cap T(W \times \dots \times W) \neq \emptyset$. But then $U_1 \times \dots \times U_n \cap T(W \times \dots \times W) \neq \emptyset$ and x is ϕ - n -ergodic.

b Let $x \in X$ be ϕ - n -ergodic, and choose $0 \subseteq (\phi^{\leftarrow} \phi(x))^n$ open in $(\phi^{\leftarrow} \phi(x))^n$. Let V_1, \dots, V_n be open in $\phi^{\leftarrow} \phi(x)$ such that $V_1 \times \dots \times V_n \subseteq 0$ and let $W_{X_0} = \{W_\alpha \mid \alpha \in \mathbb{N}\}$.

For $\alpha \in \mathbb{N}$ define t_α and $V_1^\alpha, \dots, V_n^\alpha$ inductively as follows: let $t_1 \in T$ be such that $t_1(V_1 \times \dots \times V_n) \cap W_1 \times \dots \times W_1 \neq \emptyset$ and let $V_i^1 := V_i$. Let t_α and V_i^α be defined such that $t_\alpha(V_1^\alpha \times \dots \times V_n^\alpha) \cap W_\alpha \times \dots \times W_\alpha \neq \emptyset$. Then choose $V_i^{\alpha+1} \neq \emptyset$ such that $V_i^{\alpha+1} \subseteq \overline{V_i^{\alpha+1}} \subseteq V_i^\alpha \cap t_\alpha^{-1} W_\alpha$ and let $t_{\alpha+1} \in T$ be such that $t_{\alpha+1}(V_1^{\alpha+1} \times \dots \times V_n^{\alpha+1}) \cap W_{\alpha+1} \times \dots \times W_{\alpha+1} \neq \emptyset$. Choose $x_i \in \bigcap \{V_i^\alpha \mid \alpha \in \mathbb{N}\} \subseteq V_i \cap \{t_\alpha^{-1} W_\alpha \mid \alpha \in \mathbb{N}\}$ for all $i = 1, \dots, n$ then $(x_1, \dots, x_n) \in 0$ and $t_\alpha(x_1, \dots, x_n) \rightarrow (x_0, \dots, x_0)$, i.e. x is a P_ϕ^n point. \square

Using an idea of McMAHON we will prove the existence of ϕ - n -local ergodic points under certain conditions. For that we need the following theorem.

8. THEOREM. Let $\phi: X \rightarrow Y$ be a RIM extension of minimal ttg's with section $\lambda: Y \rightarrow M(X)$ and let $x_0 \in X$.

a Let U be open in $\phi^{\leftarrow} \phi(x_0)$ and $F \subseteq \phi^{\leftarrow} \phi(x_0)$. Then

$$E_\phi[F] \times (U \cap \text{supp } \lambda_{\phi(x_0)}) \subseteq T(F \times U)$$

b Let $n \in \mathbb{N}$, $n \geq 2$, U_i open in $\text{supp } \lambda_{\phi(x_0)}$ for $i = 1, \dots, n$ such that $E_\phi[U_i]$ is open in $\phi^{\leftarrow} \phi(x_0)$ for $i = 2, \dots, n$. Then $E_\phi[U_1] \times (E_\phi[U_2] \cap \text{supp } \lambda_{\phi(x_0)}) \times \dots \times (E_\phi[U_n] \cap \text{supp } \lambda_{\phi(x_0)}) \subseteq \overline{T(U_1 \times \dots \times U_n)}$.

PROOF. a This is a special case of [M] 1.4.

b This is a special case of [MW'80] 1.1. \square

9. PROPOSITION. Let $\phi: X \rightarrow Y$ be a RIM extension of minimal ttg's with section λ . Let $\kappa: X \rightarrow X/E_\phi$ and $x \in X$. If x has a neighbourhood V in $\phi^{\leftarrow}\phi(x)$ with $E_\phi[V] \subseteq \text{supp } \lambda_{\phi(x)}$ and $\kappa' = \kappa|_{\phi^{\leftarrow}\phi(x)}: \phi^{\leftarrow}\phi(x) \rightarrow \kappa[\phi^{\leftarrow}\phi(x)]$ is open in a dense subset of V then x is ϕ -local ergodic.

PROOF. Choose $W \subseteq X$ open. Then $\kappa[W]^\circ \neq \emptyset$ (X is minimal!), so there is a neighbourhood V^* of $\kappa(x)$ and a $t \in T$ with $tV^* \subseteq \kappa[W]^\circ$. Define $U := \kappa^{\leftarrow}V^* \cap V$, and choose $n \in \mathbb{N}$, $n \geq 2$. Choose V_1, \dots, V_n open in U . Since the points of openness of κ' are dense in V and so in U we can find $V'_1 \subseteq V_1$ such that $E_\phi[V'_1]$ is open in $\phi^{\leftarrow}\phi(x)$. Obviously $E_\phi[V'_1] \subseteq E_\phi[V_1] \subseteq E_\phi[U] \subseteq E_\phi[V] \subseteq \text{supp } \lambda_{\phi(x)}$. Then by 8b we have that

$$E_\phi[V'_1] \times \dots \times E_\phi[V'_n] \subseteq \overline{T(V'_1 \times \dots \times V'_n)} \subseteq \overline{T(V_1 \times \dots \times V_n)}.$$

Since $tE_\phi[V'_1] = E_\phi[tV'_1] = \kappa^{\leftarrow}\kappa[tV'_1]$ and $\kappa[tV'_1] \subseteq tV^* \subseteq \kappa[W]^\circ$ we have that $W \cap tE_\phi[V'_1] \neq \emptyset$ and so

$$\emptyset \neq W \times \dots \times W \cap \overline{T(E_\phi[V'_1] \times \dots \times E_\phi[V'_n])} \subseteq W \times \dots \times W \cap \overline{T(V_1 \times \dots \times V_n)}$$

but then $W \times \dots \times W \cap \overline{T(V_1 \times \dots \times V_n)} \neq \emptyset$. □

10. PROPOSITION. If X is minimal and has an invariant measure μ then every $x \in X$ is local ergodic. In particular $Q_X = E_X$.

PROOF. Note that $X = \text{supp } \mu$ ($= \text{supp } \lambda_1 = \phi^{\leftarrow}\phi(x)$ with $\phi: X \rightarrow 1$). Since in the proof of Prop. 9 the openness of κ' in some points was only used to make sure that any V open in U contained a V' with $E_\phi[V']$ open in $\phi^{\leftarrow}\phi(x)$, it is enough to prove that any open V in X contains an open V' in X with $E_\phi[V']$ open. The proposition follows then as in 9.

Let $\kappa: X \rightarrow X/E_X$ and let $V' := \kappa^{\leftarrow}(\kappa[V]^\circ) \cap V$. Then $E_\phi[V'] = \kappa^{\leftarrow}\kappa[V'] = \kappa^{\leftarrow}(\kappa[V]^\circ)$ is open and non empty. From Prop. 1 it follows that $Q_X[x] = E_X[x]$ for all $x \in X$ and so $Q_X = E_X$ (which is a special case of [M]1.5). □

11. COROLLARY. a If $\phi: X \rightarrow Y$ is a RIM extension of metric minimal ttg's then there is a residual subset of ϕ -local ergodic points.

b If $\phi: X \rightarrow Y$ is a RIM extension of minimal ttg's with $R_\phi = E_\phi$ then every

$x \in X$ with $\text{supp } \lambda_{\phi(x)} = \phi^{\leftarrow} \phi(x)$ is ϕ -ergodic.

PROOF. a Let $\kappa: X \rightarrow X/E_{\phi}$. Since X is metric and minimal, there is a residual set $X_1 \subseteq X$ in each point of which κ is open and, by [G₂] 3.3, there is a residual set $X_2 \subseteq X$ with $\text{supp } \lambda_{\phi(x)} = \phi^{\leftarrow} \phi(x)$ for all $x \in X_2$. Let $X_0 = X_1 \cap X_2$ then X_0 is residual. By [V₁] prop.3.1 there is a residual subset Y_0 of Y such that $X_0 \cap \phi^{\leftarrow}(y)$ is residual in $\phi^{\leftarrow}(y)$ for all $y \in Y_0$. Choose $x \in \phi^{\leftarrow}[Y_0]$, then κ is open in a dense subset of $\phi^{\leftarrow} \phi(x)$ and so κ' is open in a dense subset of $\phi^{\leftarrow} \phi(x)$, also $\text{supp } \lambda_{\phi(x)} = \phi^{\leftarrow} \phi(x)$. So from Prop.9 it follows that x is ϕ -local ergodic. Clearly, $\phi^{\leftarrow}[Y_0]$ is residual in X .
b In this case $\kappa = \phi$. If $\text{supp } \lambda_{\phi(x)} = \phi^{\leftarrow} \phi(x)$ then $\kappa = \phi$ is open in x and so κ' is open in all $x' \in \phi^{\leftarrow} \phi(x) (= \phi^{\leftarrow} \phi(x'))$. From Prop.9 and the observation that $E_{\phi}[x] = \phi^{\leftarrow} \phi(x)$ it follows that x is ϕ -ergodic. \square

The following theorem as well as its proof is a generalization of [G₁] II.2.1.

12. THEOREM. Let $\phi: X \rightarrow Y$ and $\psi: Z \rightarrow Y$ be homomorphisms of minimal ttg's with ψ open. If for every $n \in \mathbb{N}$, $n \geq 2$ there exists a ϕ - n -ergodic point $x \in X$, then $\phi \dot{=} \psi$ (i.e. $R_{\phi\psi} = \{(x, z) \in X \times Z \mid \phi(x) = \psi(z)\}$ is ergodic.

PROOF. Choose $W = \overline{TW} \subseteq R_{\phi\psi}$ with $\text{int}_{R_{\phi\psi}} W \neq \emptyset$. Since ψ is open and X minimal we may choose open sets U and V in X and Z such that $U \times V \cap R_{\phi\psi} \subseteq W^{\circ}$ and for every $v \in V$ there is a $u \in U$ with $(u, v) \in R_{\phi\psi}$.

Since Z is minimal there are t_1, \dots, t_m in T with $Z = \bigcup_{i=1}^m t_i V$. Choose $(\bar{x}, \bar{z}) \in R_{\phi\psi}$ and a neighbourhood O of (\bar{x}, \bar{z}) in $R_{\phi\psi}$. Since ψ is open there are open neighbourhoods $O_{\bar{x}}$ and $O_{\bar{z}}$ of \bar{x} and \bar{z} in X and Z such that

$O_{\bar{x}} \times O_{\bar{z}} \cap R_{\phi\psi} \subseteq O$ and for every $o_1 \in O_{\bar{x}}$ there is an $o_2 \in O_{\bar{z}}$ with $(o_1, o_2) \in R_{\phi\psi}$.

Choose a ϕ - m -ergodic point $x \in O_{\bar{x}}$ [Note that x is ϕ - n -ergodic for all n with $2 \leq n \leq m$], and choose $z \in O_{\bar{z}}$ with $\phi(x) = \psi(z)$. Let $y := \phi(x) = \psi(z)$ and $U_x := O_{\bar{x}}$, $V_z := O_{\bar{z}}$. Without loss of generality choose $\{t_1, \dots, t_n\} \subseteq \{t_1, \dots, t_m\}$ such that $\psi^{\leftarrow}(y) \subseteq \bigcup_{i=1}^n t_i V$ and $t_i V \cap \psi^{\leftarrow}(y) \neq \emptyset$ for $i = 1, \dots, n$. Then $L := t_1 U \times \dots \times t_n U \cap (\phi^{\leftarrow}(y))^n$ is open in $(\phi^{\leftarrow}(y))^n$ and non-empty, for choose $z_i \in V \cap t_i^{-1} \psi^{\leftarrow}(y)$ and $x_i \in U$ such that $\phi(x_i) = \psi(z_i)$ then $(t_1 x_1, \dots, t_n x_n) \in L$. Since U_x is open in X and x is ϕ - n -ergodic it follows

that there exists a $t \in T$ with

$$t(t_1 U \times \dots \times t_n U \cap (\phi^{\leftarrow}(y))^n) \cap \prod_n U_x \neq \emptyset.$$

Choose $\bar{x}_i \in U$ with $tt_i \bar{x}_i \in U_x \cap \phi^{\leftarrow}(ty)$ for $i = 1, \dots, n$.

Choose $z' \in V_z$ such that $\phi(tt_i \bar{x}_i) = \psi(z') = ty$, then $t^{-1}z' \in t_{i_0} V \cap \psi^{\leftarrow}(y)$ for some $t_{i_0} \in \{t_1, \dots, t_n\}$. But now

$$(tt_{i_0} \bar{x}_{i_0}, z') \in tt_{i_0} (U \times V) \cap R_{\phi\psi} \cap U_x \times V_z \subseteq TW^0 \cap O \neq \emptyset.$$

Since O was an arbitrary neighbourhood of (\bar{x}, \bar{z}) in $R_{\phi\psi}$ it follows that $(\bar{x}, \bar{z}) \in \overline{TW} = W$ and so $R_{\phi\psi} = W$; i.e. $R_{\phi\psi}$ is ergodic. \square

13. COROLLARY. Let $\phi: X \rightarrow Y$ be an open RIM extension of minimal ttg's with section λ . Suppose there is an $x \in X$ with $\phi^{\leftarrow}\phi(x) = \text{supp } \lambda_{\phi(x)}$ (e.g. X is metric). Equivalent are

$$\left. \begin{array}{l} \underline{a} \quad Q_{\phi} = R_{\phi} \\ \underline{b} \quad E_{\phi} = R_{\phi} \end{array} \right\} \quad (\text{cf. [MW'80] Prop. 2.2})$$

$\underline{c} \quad \phi$ is weakly mixing

$\underline{d} \quad \phi \trianglelefteq \psi$ for every open homomorphism $\psi: Z \rightarrow Y$ of minimal ttg's.

PROOF. $d \Rightarrow c \Rightarrow a \Rightarrow b$ is obvious

$b \Rightarrow d$ follows from 11 and 12 and [M]2.2.. \square

14. COROLLARY. If X is minimal and has an invariant measure, then X is weakly mixing iff X is weakly disjoint from every minimal ttg (iff $E_X = Q_X = X \times X$).

15. COROLLARY. Let $\phi: X \rightarrow Y$ and $\psi: Z \rightarrow Y$ be homomorphisms of minimal ttg's with ψ open.

\underline{a} If ϕ is a RIM extension with $E_{\phi} = R_{\phi}$ and X metric then $\phi \trianglelefteq \psi$.

\underline{b} If ϕ is proximal then $\phi \trianglelefteq \psi$.

In particular any open proximal map is weakly mixing.

3. $E_\phi = Q_\phi$ AND "JOE'S CONJECTURE"

The next theorem gives a partial solution to a question raised by J. AUSLANDER. If x_1 and x_2 are regionally proximal and if we have a net $(x_i^1, x_i^2) \rightarrow (x_1, x_2)$, is it possible to find a net $(\bar{x}_i^1, \bar{x}_i^2)$, suitably close to the first one, that converges to (x_1, x_2) such that for some net $\{t_i\}$ in T $t_i(\bar{x}_i^1, \bar{x}_i^2) \rightarrow \Delta$? It is not difficult to see that we can state that question as follows: When do we have that $Q_\phi = \bigcap \{\text{int}_{R_\phi} \overline{Ta \cap R_\phi} \mid a \in U_X\}$. In the absolute case McMAHON proved the equality for minimal ttg's with invariant measure.

16. LEMMA. Let $\phi: X \rightarrow Y$ be an open RIM extension of minimal ttg's and $W \subseteq X$ open. Then there is an open set $U = E_\phi[U]$ in X such that $R_\phi \cap U_1 \times U_2 \cap T(W \times W) \neq \emptyset$ for all U_1 and U_2 open in U with $R_\phi \cap U_1 \times U_2 \neq \emptyset$.

PROOF. Let $\kappa: X \rightarrow X/E_\phi$ and define $U := \kappa^+(\kappa[W]^\circ)$. Clearly $U = E_\phi[U] = \kappa^+\kappa[U]$ is open in X .

Choose U_1 and U_2 open in U with $\phi[U_1] \cap \phi[U_2] =: V^* \neq \emptyset$. Since $U_1 \cap \phi^+(V^*) \neq \emptyset$ and open we can choose $V_1 \subseteq U_1 \cap \phi^+(V^*)$ open with $E_\phi[V_1]$ open in X .

Then $\phi[V_1] \subseteq \phi[U_2]$ and $U_2 \cap \phi^+(\phi[V_1]) \neq \emptyset$ and open. Choose $V_2 \subseteq U_2 \cap \phi^+(\phi[V_1])$ open with $E_\phi[V_2]$ open in X and remark that $\phi[V_2] \subseteq \phi[V_1]$. As $W \cap E_\phi[V_2] (\neq \emptyset)$ is open we can choose $x_0 \in W \cap E_\phi[V_2]$ with $x_0 \in \text{supp } \lambda_\phi(x_0)$, say $y_0 = \phi(x_0)$.

Clearly $y_0 \in \phi[V_2] \subseteq \phi[V_1]$ so $\tilde{V}_1 := V_1 \cap \phi^+(y_0) \neq \emptyset$ and open in $\phi^+(y_0)$ and so $E_\phi[\tilde{V}_1] = E_\phi[V_1 \cap \phi^+(y_0)] = E_\phi[V_1] \cap \phi^+(y_0)$ is open in $\phi^+(y_0)$. By Theorem 8b we have that

$$\begin{aligned} E_\phi[\tilde{V}_1] \times (E_\phi[\tilde{V}_2] \cap \text{supp } \lambda_{y_0}) &\subseteq \overline{T(\tilde{V}_1 \times \tilde{V}_2)} \subseteq \\ &\subseteq \overline{T(V_1 \times V_2 \cap \phi^+y_0 \times \phi^+y_0)} \subseteq T(U_1 \times U_2 \cap R_\phi) \end{aligned}$$

But $x_0 \in E_\phi[\tilde{V}_2] \cap \text{supp } \lambda_{y_0} \cap W$ and obviously $W \cap E_\phi[\tilde{V}_1] \neq \emptyset$ so $W \times W \cap E_\phi[\tilde{V}_1] \times (E_\phi[\tilde{V}_2] \cap \text{supp } \lambda_{y_0}) \neq \emptyset$ and $W \times W \cap T(U_1 \times U_2 \cap R_\phi) \neq \emptyset$. As $W \times W$ is open we have $W \times W \cap T(U_1 \times U_2 \cap R_\phi) = W \times W \cap T(U_1 \times U_2) \cap R_\phi \neq \emptyset$. \square

17. THEOREM. Let $\phi: X \rightarrow Y$ be an open RIM extension of minimal ttg's. Then

$$E_\phi = Q_\phi = \bigcap \{ \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in U_X \}.$$

PROOF. Choose $\alpha \in U_X$ and $\beta \in U_X$ such that $\beta = \beta^{-1}$ and $\beta \circ \beta \subseteq \alpha$. For $W := \beta(x_0)$ for some fixed $x_0 \in X$ we have

$$\overline{T(W \times W) \cap R_\phi} \subseteq \overline{T\alpha \cap R_\phi}.$$

Choose $U = E_\phi[U]$ open in X as in lemma 16.

Choose $(x_1, x_2) \in U \times U \cap R_\phi$ and open neighbourhoods V_1 and V_2 of x_1 and x_2 in U then $V_1 \times V_2 \cap R_\phi \neq \emptyset$ so by lemma 16

$$T(W \times W) \cap R_\phi \cap V_1 \times V_2 \neq \emptyset.$$

But then $(x_1, x_2) \in \overline{T(W \times W) \cap R_\phi} \subseteq \overline{T\alpha \cap R_\phi}$ and so $U \times U \cap R_\phi \subseteq \overline{T\alpha \cap R_\phi}$, even $U \times U \cap R_\phi \subseteq \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$. Choose $(x_1, x_2) \in E_\phi$ and $t \in T$ with $tx_1 \in U$.

Then $(tx_1, tx_2) \in E_\phi[tx_1] \times E_\phi[tx_2] \subseteq E_\phi[U] \times E_\phi[U] \cap R_\phi = U \times U \cap R_\phi \subseteq \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$. As $\text{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$ is T invariant we have $(x_1, x_2) \in \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$ and consequently $E_\phi \subseteq \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$.

So $E_\phi \subseteq \bigcap \{ \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in U_X \} \subseteq \bigcap \{ \overline{T\alpha \cap R_\phi} \mid \alpha \in U_X \} = Q_\phi$. \square

18. REMARK. We used the openness of ϕ to conclude that $(\phi[V_1] \cap \phi[V_2])^\circ \neq \emptyset$ for open V_1 and V_2 in X with $\phi[V_1] \cap \phi[V_2] \neq \emptyset$.

Now suppose ϕ satisfies B.c, i.e. the almost periodic points are dense in R_ϕ .

Then $V_1 \times V_2 \cap R_\phi$ contains an almost periodic point (x_1, x_2) and

$V_1 \times V_2 \cap \overline{T(x_1, x_2)}$ in an open subset of the minimal ttg $\overline{T(x_1, x_2)}$. So

$\phi[V_1] \times \phi[V_2] \cap \Delta_Y$ has a non-empty interior in Δ_Y , or what is the same $(\phi[V_1] \cap \phi[V_2])^\circ \neq \emptyset$.

19. COROLLARY. If $\phi: X \rightarrow Y$ is a RIM extension of minimal ttg's that satisfies the Bronstein condition, then

$$E_\phi = Q_\phi = \bigcap \{ \text{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in U_X \}.$$

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